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NOTE

HIGHLY CONNECTED NON-2-LINKED DIGRAPHS

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For every natural number k there exists a strongly k-connected digraph which contains two vertices through which there is no directed cycle.

Introduction

Bermond and Lovász [1,4] asked if there exists a natural number k_0 such that every strongly k_0-connected digraph contains a dicycle (directed cycle) through any two prescribed vertices. The present author made the stronger conjecture [5] that there exists a function $f(k)$ such that every strongly $f(k)$-connected digraph D is strongly k-linked, i.e. for every $2k$-tuple $x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k$ of distinct vertices in D, there exist k pairwise disjoint dipaths $P_1, P_2, \ldots, P_k$ such that P_i starts in x_i and terminates in y_i, $i = 1, 2, \ldots, k$. (For tournaments, the conjecture was verified in [6]). These and related problems are discussed in [7] (see also [2]). In this paper we show that neither k_0 nor $f(2)$ exist. Moreover, we show that the 2-linkage problem, which was shown to be NP-complete by Fortune et al [3], remains to be NP-complete even for highly connected digraphs. The terminology is the same as in [2].

The construction

We describe a planar digraph G_k as follows: First take $k+1$ pairwise disjoint dipaths $P_1, P_2, \ldots, P_{k+1}$ where $P_i : v_{1,i}v_{2,i} \ldots v_{m,i}$, $m = (2k)^i$. For each $i (1 \le i \le k)$ and each $j (1 \le j \le (2k)^i)$ we add the k arcs (directed edges) $v_{q,i+1}v_{j,i}$ where

$$2k(j-1) < q \le 2k(j-1) + k$$

and the k arcs $v_{j,i}v_{q,i+1}$ where

$$2k(j-1) + k < q \le 2kj.$$

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The resulting digraph G_k can be draw in the plane such that each P_i is a vertical straight line segment with x-coordinate i and such that all arcs are straight line segments directed upwards. Let G'_k be a copy of G_k obtained by reflecting G_k in the y-axis. If $H \subseteq G_k$ we denote the corresponding subgraph of G'_k by H'. For each vertex v in P_1 we add the arcs vv' and $v'v$ to $G_k \cup G'_k$ and denote the resulting digraph by G''_k. We let A denote those vertices of P_{k+1} from which $v_{1,2}$ can be reached by a dipath, and we let B denote those vertices of P_{k+1} that can be reached from $v_{(2k)^2,2}$ by a dipath. Since all arcs of G''_k except those between G'_k and G_k are directed upwards, we conclude that

(1) G''_k has no dipath from A to A' or from A' to A or from B to P'_{k+1} or from B' to P_{k+1} or from B to A or from B' to A'.

It is easy to verify that

(2) G_k has k disjoint dipaths from A to B.

(1) combined with the planarity of G''_k imply that,

(3) if Q_1 is a dipath from A to P'_{k+1} and Q_2 is a dipath from A' to P_{k+1}, then $Q \cap Q_2 \neq \emptyset$.

Theorem 1. *For each natural number k, there exists a non-2-linked strongly k-connected digraph D_k.*

Proof. We form the disjoint union of G''_k and four strongly k-connected digraphs H_1, H_2, H'_1, H'_2. Then we add all arcs from S to T and from S' to T' where (S,T) is any of the ordered pairs (H_1, A), (B, H_2), (H_2, P_{k+1}), (P_{k+1}, H_1).

We first claim that the resulting digraph D_k is strongly k-connected. To see this we let V be a set of less than k vertices and we shall prove that $D_k - V$ is strongly connected. Since H_1 is strongly k-connected, $H_1 - V$ belongs to a strong component D of $D_k - V$. Clearly, D_k has k disjoint dipaths from H_2 to H_1. Combining this with (2) and the assumption that $H_2 - V$ is strongly connected we conclude that $(H_2 \cup P_{k+1}) - V$ is contained in the strong component D. For each vertex v in $G_k - V$, there is in $G_k - V$ a dipath from v to P_{k+1} and one from P_{k+1} to v. Hence v is in D. So $G_k - V$ is in D. Clearly, D contains some vertex of P'_1. Repeating the above argument we conclude that $D = D_k - V$. Hence D_k is strongly k-connected.

Now let $x_1 \in V(H_1)$, $y_1 \in V(H'_2)$, $x_2 \in V(H'_1)$ and $y_2 \in V(H_2)$. We claim that D_k has no two disjoint dipaths Q_1, Q_2 such that Q_i starts at x_i and terminates in y_i for $i = 1,2$. Suppose (reductio ad absurdum) that such two Q_1 and Q_2 do exist. Let R_1 be the segment of Q_1 starting in A such that R_1 has only its first vertex in common with $A \cup H_1$. Let Q''_1 be the initial segment of R_1 such that Q''_1 has its last vertex w (and no other vertex) in $P'_{k+1} \cup B$. If $w \in B$, then R_1 contains a segment in G''_k from B to P'_{k+1}, contradicting (1). (We are here using the fact that a dipath from $P_{k+1} \setminus (A \cup B)$ to P'_{k+1} must intersect every dipath from A to B in G''_k). Hence $w \in P'_{k+1}$. In other words, Q''_1 is a dipath in G''_k from A to P'_{k+1}. Similarly, Q_2 contains a segment from A' to P_{k+1}. By (3), $Q_1 \cap Q_2 \neq \emptyset$, a contradiction.

Theorem 2. *For each natural number k, there exists a strongly k-connected digraph D'_k containing two vertices x_0, x'_0 such that D'_k has no dicycle containing x_0 and x'_0.*

Proof. Let D_k be the digraph in Theorem 1. Add x_0 and x'_0 to D_k together with all arcs from x_0 to H_1, from H_2 to x_0, from x'_0 to H'_1 and from H'_2 to x'_0. The resulting digraph D'_k has the desired properties, by Theorem 1.

Theorem 3. *For each fixed natural number k, the following problem is NP-complete: Given four vertices x_1, x_2, y_1, y_2 in a k-connected digraph D. Does D contain two disjoint dipaths Q_1, Q_2 such that Q_i goes from x_i to y_i for $i = 1, 2$.*

Proof. Fortune et al [3] proved Theorem 3 for $k = 1$. We shall reduce the general case to that special case. Consider therefore a digraph D (with no connectivity condition imposed) and with four prescribed vertices x'_1, x'_2, y'_1, y'_2. Let D_k, x_1, x_2, y_1, y_2 be as in the proof of Theorem 1. In G''_k it is easy to find a dipath Q_0 from A' to B' having precisely one arc uv in common with P_{k+1}. Let v_1 (respectively v_2) be the successor of v on P_{k+1} (respectively Q_0). Let u_1 (respectively u_2) be the predecessor of u on P_{k+1} (respectively Q_0). Now delete u and v from D_k and add instead a copy of D. Add the arcs $u_2x'_1$, $u_1x'_2$, y'_1v_1, y'_2v_2. Add also all arcs from H_2 to D and from D to H_1. Now $x_1 \in V(H_1)$, $y_1 \in V(H'_2)$, $x_2 \in V(H'_1)$, $y_2 \in V(H_2)$. Then the resulting digraph is strongly k-connected and has two disjoint dipaths from x_i to y_i $(i = 1, 2)$ iff D has two disjoint dipaths from x'_i to y'_i $(i = 1, 2)$.

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